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Statistical techniques and Doppler satellite positioning

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Two problems are addressed: (*a*) the detection of outliers in (Doppler satellite) observations, and (*b*) the testing of coordinates in (Doppler satellite) networks. In both problems, confidence regions of the 'out of context' and 'within context' varieties are developed, and it is shown that the latter are in general about $1\frac{1}{2}$ times larger than the former (conventional) confidence regions. On the basis of this comparison, it is speculated that good data and results are being erroneously rejected. Also it is demonstrated, through the use of Bonferroni's inequality, that discarding covariances among residuals and discarding cross-covariances among station coordinates each results in a confidence level being greater than $1-\alpha$, the conventionally chosen level. As a final development, a link is made not only between univariate and multivariate testing for outliers among observations but also between testing in observation space and testing in parameter space. The implications of these developments for Doppler satellite positioning are given.

1. Introduction

The accuracy of Doppler satellite positioning results has steadily improved from the initial values of several metres to the present sub-metre accuracy. As with most developments there is always room for improvement, but after acknowledging the fairly advanced state of the art of Doppler positioning, one cannot continue to expect dramatic improvements in accuracy. However, one of the things we can do is to continue to employ mathematical techniques that will help us to get a deeper understanding and a more precise interpretation of the results. This paper focuses on some of those statistical concepts and techniques that are pertinent to Doppler satellite positioning but which as yet have not had widespread usage – especially by users of the Transit system.

In detecting outliers, Pope (1976) brought to the attention of the geodetic community a refinement that should be made to the common '2σ' approach (based on the normal distribution). In that paper, he showed that (*a*) under certain circumstances, it is necessary to replace the normal distribution with the tau distribution, and (*b*) when examining a single residual 'within context' of a set of residuals, the confidence interval is significantly larger than the usual confidence interval used to examine a single residual 'out of context', i.e. by itself or without regard to the others. This paper extends the work of Pope by adding one additional case, by treating the univariate and the multivariate situations together, and by making a graphical representation and comparison of the various confidence intervals that can be used for the detection of outliers.

Another problem addressed in this paper is the consequence of neglecting correlation among the estimated residuals (e.g. Doppler residuals). This problem has plagued geodesists since the time that high speed computers made it possible to compute the full covariance matrix of the quantities involved. It is shown that this dilemma can be overcome by employing Bonferroni's inequality.

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The last problem discussed is the assessment of coordinates in a geodetic network. Patterson (1976) has discussed Bonferroni's inequality *vis-à-vis* the testing of the coordinates in a horizontal network. Here, Bonferroni's inequality is used in the testing of three-dimensional networks (e.g. Doppler satellite networks) and one-dimensional networks (e.g. Doppler levelling (Kouba 1976)). Covered in this topic is the examination of the coordinates of one point at a time (out of context approach) and the entire network simultaneously (within context approach). A comparison is then made to the conventional 'single point error ellipse approach' frequently used in geodesy.

This paper makes exclusive use of the branch of mathematical statistics called *parametric statistics*, in which it is necessary to know, or to be able to postulate, the form of the probabilistic distribution of the observations before a statistical interpretation can be given to the solution. In particular, the statistical technique to be discusssed herein requires the observations, or equivalently the residuals, to be normally distributed. Because there is ample evidence in geodesy to show that observations are often normally distributed (see, for example, Baarda 1967, 1976) and that particular histograms of Doppler residuals (see, for example, Wells 1974) closely approximate a normal distribution, we continue to exploit fully the area of parametric statistics. It should be kept in mind, however, that the less the normality requirement is fulfilled, the less valid parametric techniques become.

2. Univariate testing for outliers

Univariate analysis is understood to be the examination of the repeated measurements of the same observable (e.g. a length) as represented by a data series $l(\tau_i)$, $i = 1, N$, where τ stands for the time coordinate and N is the sample size. There are a multitude of statistical tests that can be applied to such a data series: however, we will restrict ourselves to examining the individual observations $l(\tau_i)$ (or simply written as l_i), which are considered statistically incompatible with the rest of the series. Numerous authors (e.g. Chauvenet 1871; Dixon 1962; Willke 1965; Hamilton 1967; Pope 1976) have discussed the rejection of outliers. The most straightforward and useful tests found among these works are summarized in table 1.

Represented are four situations often found in practice. They stem from whether the population mean, μ, and population variance, σ^2, are known or unknown; when unknown they are estimated by the sample mean $\bar{l}$ and sample variance s^2. The following interpretations can be made: (*a*) 'μ known' corresponds to measuring a line of known length (e.g. a calibration baseline); (*b*) 'σ^2 known' corresponds to measuring with an instrument of known accuracy; (*c*) 'μ unknown' corresponds to measuring a line of unknown length; (*d*) 'σ^2 unknown' corresponds to measuring any line with an instrument of unknown accuracy. Each of the four cases has its own statistic, y (cf. column 5, table 1). The probability statement involving each statistic y is

$$\Pr(y_{\frac{1}{2}\alpha} < y < y_{1-\frac{1}{2}\alpha}) = 1-\alpha, \tag{1}$$

where α is the significance level. The four corresponding confidence intervals are given in column 7 of table 1 (numbered 1–4). The coefficient of σ (or s), called the expansion factor C_α, is plotted in figure 1 and labelled the 'out of context' group (the label is explained immediately below).

Embedded in table 1 are the definitions of the true residual and the estimated residual. The *true residual* $r_i = l_i - \mu$ is found in the first two statistics, while the *estimated residual* $\hat{r}_i = l_i - \bar{l}$ is found in the last two statistics. It is instructive to point out that, in each case, the denominators of the four statistics are merely the standard deviations of the residuals. In the first two cases,

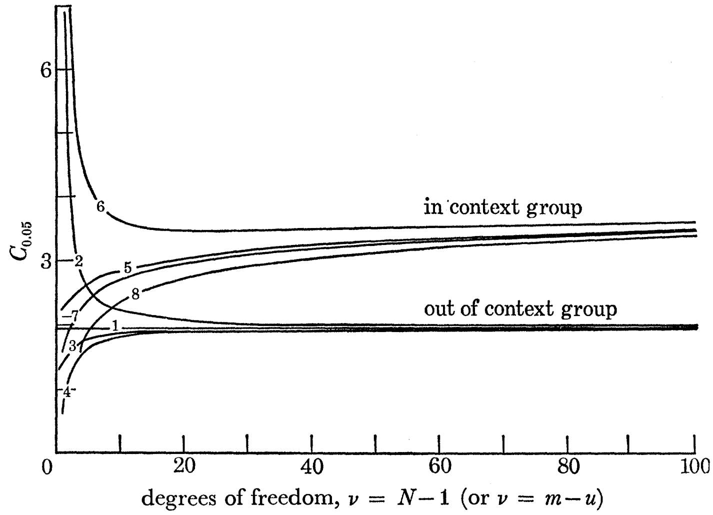

FIGURE 1. Expansion factors, $C_{0.05}$, for confidence intervals: detection of outliers.

this fact is obvious. The proof that it is true, even for the third and fourth cases, lies with the covariance matrix of the estimated residuals: namely,

$$\boldsymbol{C}_{\hat{r}} = \boldsymbol{C}_r - \boldsymbol{A}\boldsymbol{C}_{\hat{x}}\boldsymbol{A}^{\mathrm{T}}, \tag{2}$$

where $\boldsymbol{C}_r = \sigma^2 \boldsymbol{I}$ is the *a priori* covariance matrix of the residuals, the design matrix $A^{\mathrm{T}} = [1, 1, ..., 1]$, the covariance matrix (only the variance in this case) of the estimated unknowns (only the mean in this case) is $\boldsymbol{C}_{\hat{x}} \equiv \boldsymbol{C}_{\bar{l}} = \sigma^2/N$. Carrying out the algebra shows that the variance of the ith estimated residual is $(N-1)\sigma^2/N$. When σ^2 is unknown (as for the fourth statistic), the variance of the ith estimated residual is $(N-1)s^2/N$. The interesting feature of estimated residuals is that, unlike true residuals, they are correlated. For the time being, the covariance $-\sigma^2/N$ (or $-s^2/N$) need not be considered because only one residual (observation) is being examined at a time without regard to the others of the set, thereby explaining the label 'out of context'.

3. Univariate testing for outliers (within context)

In the four tests given in table 1, each l_i has actually been taken out of context, and the existence of the other individual members of the series has unwittingly been disregarded. Now the individual l_i (or r_i) will be examined within the context of their being members of the series. In doing this, it is expedient to work with the standardized residuals, i.e. the statistics y in table 1 rather than the residuals themselves. Let us first arrange them in descending order within the series. The aim is to seek cut-off points below and above which the (standardized) residuals will be rejected. The probability statement corresponding to this situation is

$$\Pr(y_{\frac{1}{2}a} < y < y_{1-\frac{1}{2}a}) = 1-a, \tag{4}$$

where y stands for any of the statistics in table 1, and a denotes a different significance level that accounts for the interplay of the individual members of the series. A convenient way to handle this problem is to derive new distribution, Y, and use the customary probability level α; however, this is not always straightforward. For the derivation of such a Y in, for example, the fourth case (table 1), see Stefansky (1972). Thompson (1935) and Pope (1976) offer a brief derivation for the general situation, an adaption of which is used here.

In the out of context approach, the selected significance level α is related to the probability of y being in $\langle y_{\frac{1}{2}\alpha}, y_{1-\frac{1}{2}\alpha}\rangle$ through (1). Realizing that all the probability distribution functions (p.d.f.s.) of table 1 are symmetrical, (1) can be rewritten as

$$1-\alpha = \Pr(|y_i| < y_{1-\frac{1}{2}\alpha}). \tag{5}$$

What we really want to do is to consider all y_i together, i.e. to consider the simultaneous probability of all y_i being within $\langle y_{\frac{1}{2}\alpha}, y_{1-\frac{1}{2}\alpha}\rangle$, or equivalently, the probability that the inequality $|y_i| < y_{1-\frac{1}{2}\alpha}$ holds simultaneously for all y_i. Denoting this *simultaneous probability* by $1-a$, we have

$$1-a = \Pr\left(\prod_{i=1}^{N}(|y_i| < y_{1-\frac{1}{2}\alpha})\right). \tag{6}$$

If the y_i are statistically independent (situations 1 and 2 of table 1), then the following equation is valid:

$$\Pr\left(\prod_{i=1}^{N}(|y_i| < y_{1-\frac{1}{2}\alpha})\right) = \prod_{i=1}^{N}\Pr(|y_i| < y_{1-\frac{1}{2}\alpha}) = \prod_{i=1}^{N}(1-\alpha) = (1-\alpha)^N. \tag{7}$$

and thus from (6)

$$1-a = (1-\alpha)^N. \tag{8}$$

We immediately see that the probability $(1-a)$ of all y_i being inside $\langle y_{\frac{1}{2}\alpha}, y_{1-\frac{1}{2}\alpha}\rangle$ is theoretically much smaller (for $(1-\alpha) < 1$) than the corresponding probability $(1-\alpha)$ for any single, isolated y_i taken out of context.

Consequently, if we want to use a prescribed significance level α when employing the within context approach, then we must use a smaller value for the significance level for the individual components; namely, from (8),

$$a \approx \alpha/N. \tag{9}$$

Rewriting (6) yields

$$\Pr\left(\prod_{i=1}^{N}(|y_i| < y_{1-\frac{1}{2}\alpha/N})\right) = \prod_{i=1}^{N}(1-\alpha/N) \approx 1-\alpha. \tag{10}$$

The smaller significance level of α/N in the above equation produces four additional tests paralleling those in table 1: these four cases are numbered 5–8 and are displayed in figure 1 (labelled the 'within context group'). Note that the latter group of tests are, for large degrees of freedom, about $1\frac{1}{2}$ times the size of the out of context group. This means that, for any given set, fewer good observations may be (erroneously) rejected when using the within context approach.

Noting that the last two statistics in table 1 contain residuals, which amongst themselves are not statistically independent, we can conclude that (6) is not exact for these two statistics. Their correlation coefficient is, however, small, namely

$$\rho_{ij} = \frac{-1/N}{\{(N-1)/N\}^{\frac{1}{2}}\{(N-1)/N\}^{\frac{1}{2}}} = -\frac{1}{(N-1)}. \tag{11}$$

In any case, we can be protected from these correlations by using Bonferroni's inequality (see, for example, Miller 1966; Feller 1968), which tells us that whether or not $y_1, y_2, \ldots, y_N$ are correlated,

$$\Pr\left(\prod_{i=1}^{N}(|y_i| < y_{1-\frac{1}{2}\alpha/N})\right) \geqslant 1-\sum_{i=1}^{N}\alpha/N = 1-\alpha; \tag{12}$$

Table 1. Testing for outliers (univariate case)

(θ_1 and θ_2 denote parameters of the univariate probability density function (p.d.f.).)

name	situation θ_1	situation θ_2	H_0 (null hypothexis)	statistic y	p.d.f.† $\phi(y)$	$1-\alpha$‡ confidence interval for the quantity tested	remarks
1 normal test of a single observation 5	μ known	σ^2 known	l_i belongs to a sample with histogram compatible with the postulated $n(l; \mu, \sigma^2)$	$(l-\mu)/\sigma$	standard normal $n(0, 1)$	$\mu-\sigma\, n_{1-\frac{1}{2}\alpha} < l_i < \mu+\sigma\, n_{1-\frac{1}{2}\alpha}$	σ^2 known thus the normal distribution
2 Student's t test of a single observation 6	μ known	σ^2 unknown; s^2 used		$(l-\mu)/s$	Student's t t_{N-1}	$\mu-s\, t_{N-1,1-\frac{1}{2}\alpha} < l_i < \mu+s\, t_{N-1,1-\frac{1}{2}\alpha}$	σ^2 estimated by s^2 without use of l
3 normal test of a single observation 7	μ unknown; $\bar{l}$ used	σ^2 known		$\dfrac{l-\bar{l}}{\{(N-1)/N\}^{\frac{1}{2}}\sigma}$	standard normal $n(0, 1)$	$\bar{l}-\{(N-1)/N\}^{\frac{1}{2}}\sigma\, n_{1-\frac{1}{2}\alpha} < l_i < \bar{l}+\{(N-1)/N\}^{\frac{1}{2}}\sigma\, n_{1-\frac{1}{2}\alpha}$	σ^2 known thus the normal distribution
4 τ test of a single observation 8	μ unknown; $\bar{l}$ used	σ^2 unknown; s^2 used		$\dfrac{l-\bar{l}}{\{(N-1)/N\}^{\frac{1}{2}}s}$	tau τ_{N-1}	$\bar{l}-\{(N-1)/N\}^{\frac{1}{2}}s\, \tau_{N-1,1-\frac{1}{2}\alpha} < l_i < \bar{l}+\{(N-1)/N\}^{\frac{1}{2}}s\tau_{N-1,1-\frac{1}{2}\alpha}$	$\bar{l}$ and s^2 computed from the same sample thus the τ distribution

† Pope (1976): $n(0, 1)$ = standard normal distribution of 0 mean and variance 1; t_{N-1} = Student's t distribution with $N-1$ degrees of freedom; τ_{N-1} = tau distribution with $N-1$ degrees of freedom.

‡ To test within context of the series: replace α in the above with $a = \alpha/N$, where N is the number of members in the series (numbered 5–8 in figure 1).

Table 2. Testing for outliers (multivariate case)

($\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ denote parameters of the multivariate p.d.f.)

name	situation $\boldsymbol{\theta}_1$	situation $\boldsymbol{\theta}_2$	situation σ_0^2	H_0 (null hypothesis)	statistic y	p.d.f.† ϕ_y	$1-\alpha$‡ confidence interval	remarks
test of a residual outlier	$\boldsymbol{\mu} = \mathbf{0}$ unknown	$\boldsymbol{C}_{\hat{r}}$ known	σ_0^2 known	$\hat{r}_i$ belongs to a sample with histogram compatible with the postulated $n(\hat{r}_i; 0, \sigma^2_{\hat{r}_i})$	$\tilde{r}_i = \hat{r}_i/\sigma_{\hat{r}_i}$	standard normal $n(\tilde{r}; 0, 1)$	$n_{\frac{1}{2}\alpha}\sigma_{\hat{r}_i} < \hat{r}_i < n_{1-\frac{1}{2}\alpha}\sigma_{\hat{r}_i}$	$\hat{r}_i$ taken from $\mathcal{N}(\hat{\boldsymbol{r}}; \mathbf{0}, \boldsymbol{C}_{\hat{r}})$
	$\boldsymbol{\mu} = \mathbf{0}$ unknown	$\hat{\boldsymbol{C}}_{\hat{r}}$	unknown σ_0^2		$\tilde{r}_i = r_i/\hat{\sigma}_{\hat{r}_i}$	tau τ_ν	$\tau_{\nu,\frac{1}{2}\alpha}\hat{\sigma}_{\hat{r}_i} < \hat{r}_i < \tau_{\nu,1-\frac{1}{2}\alpha}\hat{\sigma}_{\hat{r}_i}$	$\nu = m-u$ is the redundancy of the adjustment; m = number of equations; u = number of unknowns.

† Pope (1976): τ_ν denotes the τ distribution with ν degrees of freedom.

‡ To test the vector $\hat{r}$ 'within context', replace α with $a = \alpha/n$, where n is the dimension of $\hat{r}$.

this means that the probability is at least as large as $1-\alpha$. Thus, Bonferroni's inequality solves, rather simply, the worrisome problem of what to do after correlation has been neglected.

4. Multivariate testing for outliers

Up to this point, we have been preoccupied with only one element l_i, and its repeated measurements (N), in the vector of observations $\boldsymbol{l}^{\mathrm{T}} = [l_1, l_2, ..., l_i, ..., l_n]$. The simultaneous assessment of several or all n members of the vector is called *multivariate analysis*. The objective of multivariate analysis is to determine how well the different observations l_i (e.g. Doppler satellite measurements) fit a mathematical model. Another motivation for using multivariate analysis is that it allows the correlated residuals of the univariate case to be treated formally, which is an impossibility within a univariate context with the use of merely a univariate probability density function (p.d.f.). A multivariate p.d.f. employing the vector of residuals $\hat{\boldsymbol{r}}$ and its corresponding covariance matrix $\boldsymbol{C}_{\hat{r}}$ (e.g. equation (2)) is needed for this purpose.

In the real world of experimentation, we have only an estimate $\hat{\boldsymbol{l}}$ of the vector of observations instead of the 'true' vector $\boldsymbol{\mu}$. This means that the 'true' value of the residual vector $\boldsymbol{l}-\boldsymbol{\mu} = \boldsymbol{r}$ must be replaced by $\boldsymbol{l}-\hat{\boldsymbol{l}} = \hat{\boldsymbol{r}}$, the estimated value. Accordingly, the 'true' covariance matrix $\boldsymbol{C}$, corresponding to $\boldsymbol{\mu}$, must also be considered unknown. Thus, instead of characterizing the observations with the multivariate normal distribution $\mathscr{N}(\boldsymbol{l};\ \boldsymbol{\mu},\ \boldsymbol{C})$, or equivalently $\mathscr{N}(\boldsymbol{l}-\boldsymbol{\mu};\ \boldsymbol{0},\ \boldsymbol{C}) = \mathscr{N}(\boldsymbol{r};\ \boldsymbol{0},\ \boldsymbol{C})$, we choose to use the estimated residual vector $\hat{\boldsymbol{r}}$ and its covariance matrix $\boldsymbol{C}_{\hat{r}}$, which is denoted by $\hat{\boldsymbol{C}}_{\hat{r}}$ if its scale (variance factor $\hat{\sigma}_0^2$) is also estimated; i.e.

$$\phi_l = \mathscr{N}(\boldsymbol{l};\hat{\boldsymbol{l}},\boldsymbol{C}_{\hat{l}}) = \mathscr{N}(\boldsymbol{l}-\hat{\boldsymbol{l}};\boldsymbol{0},\boldsymbol{C}_{l-\hat{l}}) = \mathscr{N}(\hat{\boldsymbol{r}};\boldsymbol{0},\boldsymbol{C}_{\hat{r}}) = \hat{K}^{-1}\exp[-\tfrac{1}{2}\hat{\boldsymbol{r}}^{\mathrm{T}}\boldsymbol{C}_{\hat{r}}^{-1}\hat{\boldsymbol{r}}], \tag{13}$$

where

$$\hat{K} = (2\pi)^{\frac{1}{2}n}(\det \boldsymbol{C}_{\hat{r}})^{\frac{1}{2}}.$$

The one main problem with the above p.d.f. that stops us from proceeding any further is the fact that the regular inverse for $\boldsymbol{C}_{\hat{r}}$ does not exist because its rank is less than its dimension. It is not at all clear that the use of generalized inverses here would lead to practical results.

Another route usually taken to circumvent the problem is to discard the covariance elements (as was done in the univariate case). This action allows us to proceed and standardize each residual $\hat{r}_i$, $i = 1, n$, i.e.

$$\tilde{r}_i = \hat{r}_i/\sigma_{\hat{r}_i} = (l_i-\hat{l}_i)/\sigma_{l_i-\hat{l}_i}, \tag{14}$$

where $\sigma_{\hat{r}_i}$ is the square root of the ith diagonal element of $\boldsymbol{C}_{\hat{r}}$. After pooling, we can produce a standard univariate normal p.d.f. So far we have tacitly assumed that the scale of $\boldsymbol{C}_r$ was known. In the case where it is unknown, the standardization takes the form

$$\tilde{r}_i = \hat{r}_i/\hat{\sigma}_{\hat{r}_i}, \tag{15}$$

where $\hat{\sigma}_{\hat{r}_i}$ is the estimated standard deviation. Equations (14) and (15) form the basis of the two tests that can be employed when searching for outliers in the multivariate case (cf. table 2). Again, as in the univariate case, the residuals can be examined out of context or within context. The expansion factor for the first test in table 2 is equivalent to the univariate case – numbered 1 and 5 in table 1. The only difference between the second test in table 2 and its univariate counterpart (numbered 4 and 8 in table 1) is the factor $\{(N-1)/N\}^{\frac{1}{2}}$. Thus figure 1 can be used to study the expansion factors for the multivariate case. If exact values are desired, they should be computed with the use of the τ distribution or obtained from special tables, e.g. those of Pope (1976).

5. Multivariate testing of coordinates

After having screened the observations for outliers (§§ 2–4) and examined the mathematical model through the variance factor test (see, for example, Mikhail 1976), we can now turn to the assessment of the parameters $\boldsymbol{x}$, e.g. the tracking station coordinates as determined from a least squares solution. Since the p.d.f. of the observations has been postulated to be multivariate normal (cf. equation (13)), and given the fact that the estimated parameters $\boldsymbol{x}$ are a linear combination of these, it is well known from statistics that the p.d.f. of the parameters will also be multivariate normal (see, for example, Hamilton 1967). Because in the real world of experimentation we can only get the estimate $\hat{\boldsymbol{x}}$ of $\boldsymbol{\mu}_x$ and $\boldsymbol{C}_{\hat{x}}$ of $\boldsymbol{C}_x$, the p.d.f. of concern is

$$\begin{aligned}\phi_x &= \mathcal{N}(\boldsymbol{x};\, \hat{\boldsymbol{x}},\, \boldsymbol{C}_{\hat{x}}) \\ &= \hat{K}^{-1}\exp[-\tfrac{1}{2}(\boldsymbol{x}-\hat{\boldsymbol{x}})^{\mathrm{T}}\boldsymbol{C}_{\hat{x}}^{-1}(\boldsymbol{x}-\hat{\boldsymbol{x}})],\end{aligned} \tag{16}$$

where $\hat{K} = (2\pi)^{\frac{1}{2}u}\,(\det \boldsymbol{C}_{\hat{x}})^{\frac{1}{2}}$. In the above, the vector $\hat{\boldsymbol{x}}$ is assumed to be a stochastic quantity, the u components of which are correlated. The quadratic form in the above p.d.f. is the basis for testing.

The two multivariate tests of the parameters treated herein correspond to the cases in which the variance factor σ_0^2 is known or unknown, i.e. we use either $\boldsymbol{C}_{\hat{x}}$ or $\hat{\boldsymbol{C}}_{\hat{x}}$. Considering that σ_0^2 is known (i.e. we know $\boldsymbol{C}_l$ and thus we have $\boldsymbol{C}_{\hat{x}}$), the statistic used is

$$y = (\boldsymbol{x}-\hat{\boldsymbol{x}})^{\mathrm{T}}\boldsymbol{C}_{\hat{x}}^{-1}(\boldsymbol{x}-\hat{\boldsymbol{x}}). \tag{17}$$

Its p.d.f. is χ_u^2 with u degrees of freedom since it is assumed that the difference vector $(\boldsymbol{x}-\hat{\boldsymbol{x}})$ is a random variable which is normally distributed (cf. Graybill 1976). On the other hand, considering that σ_0^2 is unknown (i.e. $\boldsymbol{C}_l$ is known up to a scalar factor and thus only $\hat{\boldsymbol{C}}_{\hat{x}}$ is available), the statistic used is

$$y = \frac{(\boldsymbol{x}-\hat{\boldsymbol{x}})^{\mathrm{T}}\hat{\boldsymbol{C}}_{\hat{x}}^{-1}(\boldsymbol{x}-\hat{\boldsymbol{x}})}{u}. \tag{18}$$

This is the ratio of two independent χ^2 random variables divided by their degrees of freedom and thus this ratio has a Fisher ($F_{u,\,m-u}$) distribution (cf. Hogg & Craig 1970). (Note that $m-u = \nu$ is the redundancy in the solution.) These two tests are used to check the compatibility between a hypothesized set of values for $\boldsymbol{x}$, and $\hat{\boldsymbol{x}}$. They are both rigorous and clearly take into account all the variances and covariances.

One runs into difficulty, however, when only subsets $\boldsymbol{x}_k$ of the vector $\boldsymbol{x}$ are examined individually, which is often what we want to do in practice. To investigate this situation, let us write our quadratic form (17) as $\Delta\boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}_{\hat{x}}^{-1}\Delta\boldsymbol{x}$. Denote the k subset of the vector $\boldsymbol{x}$ by $\boldsymbol{x}_k$ and its corresponding quadratic form by $\Delta\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{C}_{\hat{x}_k}^{-1}\Delta\boldsymbol{x}_k$, where $\boldsymbol{C}_{\hat{x}_k}^{-1}$ is the appropriate submatrix of $\boldsymbol{C}_{\hat{x}}^{-1}$. The probability statement for such a subset can be written as

$$\Pr(\Delta\boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{C}_{\hat{x}_k}^{-1}\Delta\boldsymbol{x}_k \leqslant y_{1-\alpha}) = 1-\alpha, \tag{19}$$

where $y_{1-\alpha}$ is determined from the p.d.f. belonging to this statistic. It is $\chi^2_{q,\,1-\alpha}$ with q degrees of freedom, where q is the dimension of $\boldsymbol{x}_k$.

On the other hand, the simultaneous probability of the quadratic form of all the N subvectors being smaller than $y_{1-\alpha}$ (with $\phi_y = \chi^2_{q,1-\alpha}$) is significantly smaller (analogous to (8)). Again we can use Bonferroni's inequality and rewrite the above as follows:

$$\Pr\left(\prod_{k=1}^{N} \Delta \boldsymbol{x}_k^{\mathrm{T}} C_{\hat{\boldsymbol{x}}_k}^{-1} \Delta \boldsymbol{x}_k \leqslant y_{1-\alpha}\right) \geqslant 1 - \sum_{k=1}^{N} \alpha = 1 - N\alpha. \tag{20}$$

It directly shows that if the probability $1-\alpha$ is desired for the simultaneous (within context) formulation, then the size of the individual confidence regions of the subvectors must be increased so as to correspond to the change from $y_{1-\alpha}$ to $y_{1-\alpha/N}$, i.e. (20) becomes

$$\Pr\left(\prod_{k=1}^{N} \Delta \boldsymbol{x}_k^{\mathrm{T}} C_{\hat{\boldsymbol{x}}_k}^{-1} \Delta \boldsymbol{x}_k \leqslant y_{1-\alpha/N}\right) \geqslant 1 - \sum_{k=1}^{N} \alpha/N = 1 - \alpha. \tag{21}$$

For the case of σ_0^2 being unknown and replaced by $\hat{\sigma}_0^2$, the statistic y in (21) has the p.d.f. $qF_{q,m-u}$. We note that the inequality ($\geqslant 1-\alpha$) is considered to be the result of neglecting the cross-covariances between the subvectors $\boldsymbol{x}_k$ when formulating the simultaneous probability statement. It is important to note that the investigated simultaneous (within context) probability is at least $1-\alpha$, but it is most probably larger because information, in the form of cross-covariances, has been neglected leaving us with a conservative estimate.

Let us now specialize the above to the case $q = 1$, i.e. subvectors of one element (which means individual components of $\boldsymbol{x}$). We can write

$$\Pr\left(\prod_{k=1}^{N} \Delta \boldsymbol{x}_k^2 / \hat{\sigma}_{\hat{\boldsymbol{x}}_k}^2 \leqslant y_{1-\alpha/N}\right) \geqslant 1 - \alpha, \tag{22}$$

and for $N = 1$ (out of context case)

$$\Pr(0 \leqslant |\Delta \boldsymbol{x}_k| \leqslant \hat{\sigma}_{\hat{x}_k} y_{1-\alpha}^{\frac{1}{2}}) = 1 - \alpha, \tag{23}$$

where y has the p.d.f. $F_{1,m-u}$. We see immediately that the square root of the y statistic is the factor C_α required to multiply the standard confidence interval (e.g. $\langle -\hat{\sigma}_{\hat{x}_k}, \hat{\sigma}_{\hat{x}_k} \rangle$) to obtain the $1-\alpha$ confidence interval. Denoting this factor by C_α for the desired significance level α, we obtain, for $N = 1$:

$$0 \leqslant |x_k - \hat{x}_k| \leqslant C_\alpha \hat{\sigma}_{\hat{x}_k}. \tag{24}$$

Clearly the concept of multiplying by the *expansion factor* C_α can be extended to any dimensionality q for any number of components N:

$$C_\alpha = (\chi^2_{q,1-\alpha/N})^{\frac{1}{2}}, \quad \sigma_0^2 \text{ known}; \tag{25}$$

$$C_\alpha = (qF_{q,m-u,1-\alpha/N})^{\frac{1}{2}}, \quad \sigma_0^2 \text{ unknown}. \tag{26}$$

Plotted in figure 2 are the expansion factors $C_{0.05}$ for $q = 1$. It can be seen that for growing degrees of freedom, ν, the plotted values of the F distribution approach the corresponding values based on the χ^2 distribution.

It is interesting to compare this unidimensional case (figure 2) with figure 1 (rejection of outliers). We see that the curve for $N = 1$ (out of context) corresponds exactly to the curve denoted by 2 (out of context), thus establishing a link between the testing of unidimensional quantities in parameter space and the testing of single quantities in observation space. The corresponding two 'within context' approaches show only an approximate agreement: compare curve 6 in figure 1 with curves $N = 30$ or 50 in figure 2.

It is revealing to apply the above results to geodetic networks of dimensionality $q = 1$ (e.g. height networks), $q = 2$ (e.g. horizontal networks), and $q = 3$ (satellite networks). As already noted, when the quadratic form in (19) is set equal to a constant, e.g. $y_{1-\alpha}$, we have an equation

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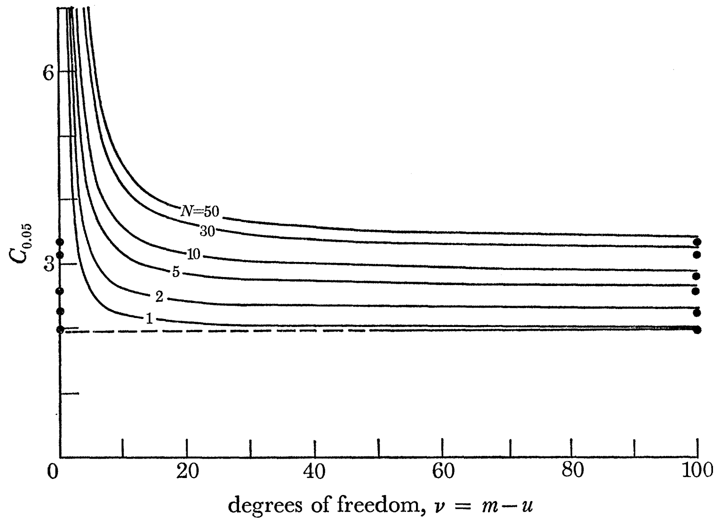

FIGURE 2. Expansion factors, $C_{0.05}$, for confidence intervals. •, χ^2; ——, F.

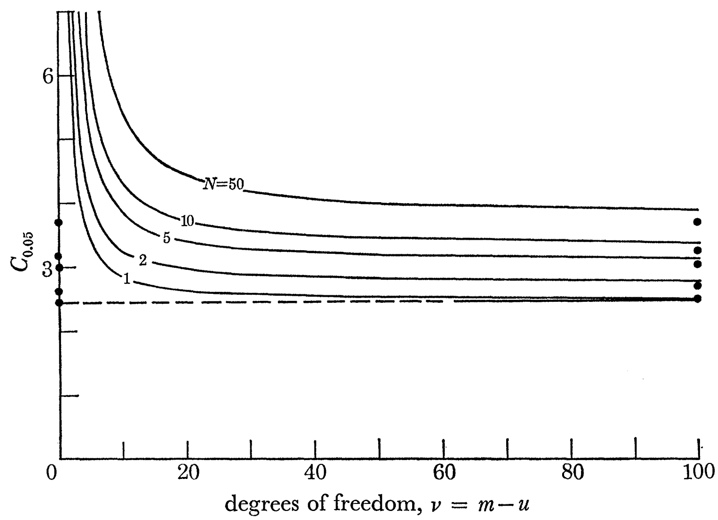

FIGURE 3. Expansion factors, $C_{0.05}$, for confidence ellipses. •, χ^2; ——, F.

of a hyperellipsoid in q dimensions centred at $\hat{\hat{x}}_k$. Figures 2, 3, and 4 show respectively the expansion factors $C_{0.05}$ for confidence intervals, ellipses and ellipsoids. For example, for σ_0^2 known, $q = 2$, $N = 1$ and $C_{0.05} = 2.4$, while for a network of points, $N = 50$ and $C_{0.05} = 3.7$, which, again, demonstrates that confidence regions (ellipses) of the within context variety are about $1\frac{1}{2}$ times those of the out of context variety.

6. Implications for Doppler satellite positioning

(*a*) In testing Doppler residuals for outliers, the entire set (e.g. for a pass) should be taken as a unit and each residual examined within the context of the others by using one of the multivariate tests given in table 2.

(*b*) After discarding the covariances among the Doppler residuals, Bonferroni's inequality should be used to show that the confidence level is in fact greater than $1 - \alpha$.

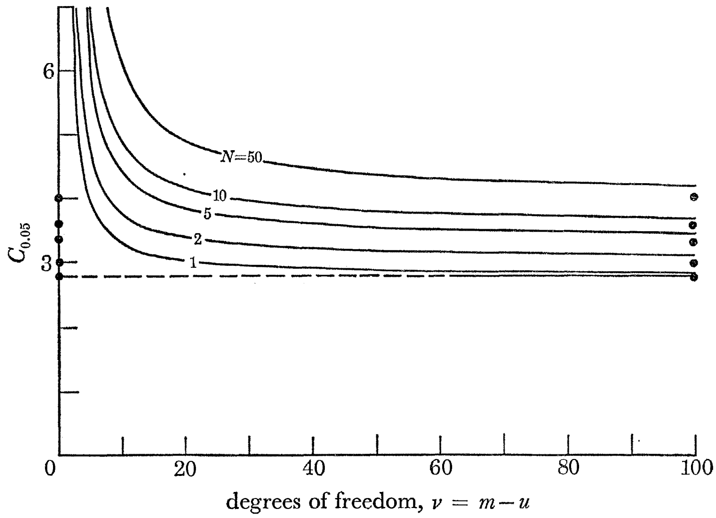

FIGURE 4. Expansion factors, $C_{0.05}$, for confidence ellipsoids. •, χ^2; ——, F.

(*c*) In testing two or more sets of Doppler coordinates for the same network, use the tests given by the quadratic forms (equations (17) and (18)). When graphical displays are used (e.g. confidence ellipses or ellipsoids), it is again necessary to use expansion factors corresponding to the within context (equations (25) and (26)) as opposed to the conventional expansion factor (out of context approach) that examines only one point at a time without regard to the others.

(*d*) Again, after discarding the cross-covariances among the points, Bonferroni's inequality should be used to show that the confidence level is in fact greater than $1-\alpha$.

(*e*) Further research is needed into mathematics for a theorem that would allow one to compute, using the known covariance elements, by how much the confidence level is greater than $1-\alpha$.

(*f*) Further research is needed to obtain a compromise value for the standard deviation of the Doppler residuals, without having to compute all the diagonal elements of the covariance matrix of the Doppler residuals, much in the same way as Pope (1976) has done for terrestrial observables.

(*g*) It would be useful to investigate the growing field of robust statistics, which is still in the realm of parametric statistics, but which is insensitive to distribution assumptions (see, for example, Huber 1964). When parametric techniques are no longer valid (because the assumption of normality is not sufficiently fulfilled), then non-parametric statistics should be exploited (see, for example, Savage (1953) for a comprehensive bibliography on the subject, and the textbook by Siegel (1956)).

This paper is essentially a selection of topics from Vaníček & Krakiwsky (1980) adapted to Doppler satellite positioning. A detailed critique of this material by G. Blaha helped in preparing the paper. Valuable discussions and exchanges of information with A. Pope, W. Knight and D. B. Thomson also contributed a great deal. R. Steeves, a graduate student at the University of New Brunswick, performed the computations and constructed the graphs. W. Wells, M. Anderson and J. Mureika produced the manuscript.

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